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COMMUTATIVE SYMMETRIC RINGS AND THEIR  
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**The Louisiana State University and Agricultural and  
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COMMUTATIVE SYMMETRIC RINGS  
AND THEIR APPLICATIONS  
TO SPECTRAL MULTIPLICITY

A Dissertation

Submitted to the Graduate Faculty of the  
Louisiana State University and  
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in

The Department of Mathematics

by

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## ABSTRACT

Weakly closed, commutative, symmetric rings of operators on a Hilbert space are examined and their structure is exploited to characterize normal operators in terms of the rings they generate.

In the first chapter, notation is established, and the less common theorems from Hilbert space theory not to be used in later sections are briefly reviewed.

In Chapter II, the structure of weakly closed, commutative, symmetric rings with a cyclic vector is specified. This theorem is then used in conjunction with a structure called a canonical decomposition system to establish that every weakly closed, commutative, symmetric ring of operators acting on a separable space and containing the identity is spatially isomorphic to a "direct sum" of  $L$  rings of a certain type.

In Chapter III, two sets of necessary and sufficient conditions are given for two normal operators on a separable Hilbert space to be unitarily equivalent. The concept of a spectral class of a normal operator is introduced - a natural generalization of the concept of an eigenvalue. Each spectral class is assigned a multiplicity, and it is shown that two normal operators are unitarily equivalent iff they have the same spectral classes with the same respective multiplicities. A multiplicity is

also defined for weakly continuous positive functionals on the symmetric ring generated by a normal operator in the weak topology, and it is shown that this multiplicity function determines the operator. This multiplicity theory is compared with a previous theory.

## CHAPTER I

### INTRODUCTION

We shall be concerned in this paper with rings of operators on a Hilbert space, in particular, weakly closed, commutative, symmetric rings. All Hilbert spaces to be considered will be complex Hilbert spaces. If  $H$  is a Hilbert space and  $x, y \in H$ , then  $(x|y)$  will denote the inner product of  $x$  and  $y$ , and  $\|x\|$  will denote the norm of  $x$ .

We shall consider only bounded linear operators in this paper, and therefore we shall use the word "operator" solely to mean a bounded linear operator. The set of all operators on the Hilbert space  $H$  will be denoted by  $B(H)$ . If  $A \in B(H)$ , then  $A^*$  will denote the adjoint of  $A$ . We employ the usual concepts of isometric, unitary, hermitian, normal, and positive definite operators.

Knowledge of the weak, strong, strongest, and norm topologies for  $B(H)$  is assumed. The norm topology is the strongest of the four; hence, any set of operators closed in one of the topologies is closed in the norm topology.

By a ring of operators we shall mean a set of operators forming a ring in the usual algebraic sense which is also a vector space over the complex numbers. A ring  $R$  of operators is symmetric if  $A^* \in R$  whenever  $A \in R$ . From the theory of Banach algebras, we know that any norm closed,



commutative, symmetric ring  $R \subseteq B(H)$  containing the identity is isometrically isomorphic to the ring of all continuous functions on the maximal ideal space of  $R$ . This isometric isomorphism - called the Gelfand transform - will be denoted by  $A \rightarrow \hat{A}(m)$ . It has the additional property that  $A^* \rightarrow \overline{\hat{A}(m)} = \text{the conjugate of } \hat{A}(m)$ .

If  $E \subseteq B(H)$  is a ring and  $\xi \in H$ , then  $\{A\xi | A \in E\}$  is a linear manifold. By  $E\xi$  we shall mean the closure of that manifold. If  $H = E\xi$ , we shall say that  $\xi$  is cyclic for  $E$  and that  $E$  has a cyclic vector. If  $E$  is a weakly closed, commutative, symmetric ring with a cyclic vector, then  $E$  is maximal in the set of all commutative, symmetric rings. If  $H$  is separable, then the converse also holds - that is, if  $E$  is a maximal commutative, symmetric ring, then  $E$  has a cyclic vector. If  $H$  is separable and  $E$  is a commutative symmetric ring, then clearly  $E'$ , the commutant of  $E$ , has a cyclic vector.

The techniques of measure and integration play an important role in this work. We employ the usual notions of  $L_1$ ,  $L_2$ , and  $L_\infty$  and adopt the near universal practise of referring to the elements of these spaces as functions, with the convention that  $f = g$  means  $f(x) = g(x)$  for every  $x$  outside some set of measure zero. If  $X$  is a compact Hausdorff space and  $\mu$  is a measure on  $X$ , then  $L_2(X, \mu)$  is a Hilbert space, and we may consider  $L_\infty(X, \mu)$  to be a ring of operators acting on  $L_2$  as follows. For each  $a \in L$

we define an operator  $A_\alpha \in B(L_2)$  by  $A_\alpha f = \alpha f$ . Note that two  $L_\infty$  functions which differ only on a set of measure zero give rise to the same operator. The ring of operators thus obtained is a maximal, commutative, symmetric ring.

If  $E$  is a weakly closed, commutative, symmetric ring with maximal ideal space  $M$ , then the fact that  $E$  is closed under sups gives rise to some strong topological properties of  $M$ . In this case,  $M$  is totally disconnected, and the closure of every open set is open. Thus,  $M$  contains many sets which are both open and closed, and such sets we call clopen. It is easy to see that projections in  $E$  correspond via the Gelfand transform to the characteristic functions of clopen sets. If  $U$  is a clopen subset of  $M$ , then we will let  $P_U$  denote the corresponding projection in  $E$ .

$\hat{P}_U(m) = \pi_U(m) =$  the characteristic function of  $U$ .

Throughout the remainder of this discussion,  $E$  will denote a weakly closed commutative, symmetric subring of  $B(H)$  with identity, and  $M$  will denote the maximal ideal space of  $E$ . Corresponding to each vector  $\xi \in H$ , there is a measure  $\mu$  on  $M$  defined by

$$(A\xi, \xi) = \int_M A(m) d\mu(m).$$

The topological properties of  $M$  carry strong implications for the resulting measure space. If  $S \subseteq M$  is measurable, then there exists a clopen set  $U$  such that  $\mu(S \setminus U \cup U \setminus S) = 0$ . If  $\xi$  is cyclic for  $E'$ , then  $\mu$  is supported on all of  $M$ , in

which case there is exactly one clopen set in each equivalence class of measurable sets. It follows that any bounded measurable function  $f(m)$  is equal almost everywhere to a continuous function, so that for some  $A \in E$ ,  $f(m)$  is equivalent to  $\hat{A}(m)$ . Similarly, each real function in  $L_2(M, \mu)$  or  $L_1(M, \mu)$  is equal almost everywhere to a continuous function from  $M$  to the extended reals.

If  $E$  has a unit cyclic vector  $\xi$  and  $\mu$  is the measure corresponding to  $\xi$ , then the map  $A\xi \rightarrow \hat{A}(m)$  can be extended to an isometry from  $H$  to  $L_2(M, \mu)$  which takes  $E$  to  $L_\infty(M, \mu)$ . Thus, any weakly closed, commutative, symmetric ring with a cyclic vector may be considered to be an  $L_\infty$  ring acting on an  $L_2$  space via multiplication. If  $U \subseteq M$  is clopen, then  $E_U = \{A|_{P_U(H)} : A \in E\}$  is a weakly closed, commutative symmetric ring with maximal ideal space  $U$ . If  $E$  has a cyclic vector, then  $E_U$  may be thought of as  $L_\infty(U)$  acting on  $L_2(U)$ .

Thus we have a very nice representation for  $E$  if  $E$  has a cyclic vector. There is also a powerful theory available when  $E'$  has a cyclic vector - which, of course, it must if  $H$  is separable. This theory was developed by Pedersen in [5] and centers around a structure called a canonical decomposition system.

Suppose  $E'$  has a unit cyclic vector  $\xi_0$  and  $\mu$  is the measure on  $M$  corresponding to  $\xi_0$ . It is demonstrated in [6] that the set of functionals on  $E$  continuous in the

weak topology coincides with the set of functionals continuous in the strongest topology. We denote this set of functionals by  $E_*$ . If  $T \in E_*$  and  $T \geq 0$ , then  $T$  has the form

$$T(A) = T_\varphi(A) = \int_M \hat{A}(m) \varphi(m) d\mu(m)$$

where  $\varphi$  is a continuous  $L_1$  function from  $M$  to the non-negative extended reals. In particular, if  $\xi \in H$ , then there exists  $\varphi_\xi \in L_1(M, \mu)$ ,  $\varphi_\xi \geq 0$ , such that

$$(A\xi, \xi) = \int_M \hat{A}(m) \varphi_\xi(m) d\mu(m)$$

for all  $A \in E$ . We denote by  $S_\xi$  the closure of  $\{m | \varphi_\xi(m) > 0\}$ .

Then  $S_\xi$  is clopen.  $P_{S_\xi}(H) = E'\xi$ .

A canonical decomposition system for  $E$  is a collection of ordered pairs  $\{(K_\alpha, \eta_\alpha)\}_{\alpha \in \Omega}$  satisfying the following.

1.  $\Omega$  is well ordered
2.  $K_\alpha = E\eta_\alpha$  and  $H = \sum_{\alpha \in \Omega} \oplus K_\alpha$
3.  $\varphi_{\eta_\alpha} = \pi_{S_{\eta_\alpha}}$
4. If  $\alpha > \beta$  then  $S_{\eta_\alpha} \subseteq S_{\eta_\beta}$ .

It can be shown that a canonical decomposition system will always exist and can be chosen to satisfy certain restrictions.

Let  $\dim_E H$  denote the smallest cardinal number  $\beta$  such that there exists a decomposition  $H = \sum_{\alpha \in \Omega} \oplus H_\alpha$  such that

each  $H_\alpha$  is cyclic for  $E$  and  $\text{card } \Omega = \beta$ . In the definition of a canonical decomposition system, the set  $\Omega$  can be chosen so that  $\text{card } \Omega = \dim_E H$ . In addition, if  $\dim_E H$  is infinite, then  $\Omega$  can be selected so that every initial segment of  $\Omega$  has smaller cardinality than  $\Omega$ . In this paper, it is to be understood that all canonical decomposition systems satisfy these restrictions.

If  $H$  is separable, then  $\dim_E H$  is countable, and  $\Omega$  will be a finite or denumerable sequence. In this case the  $S_{\eta_i}$ 's are uniquely determined by  $E$ ; they do not depend on the choice of the cyclic vector for  $E$  or the  $\eta_i$ 's. Furthermore, they can be given a characterization in terms of  $M$ . Let

$$d(m) = \text{least } \{\dim_{E_U} P_U(H) : U \subseteq M, U \text{ clopen}, \\ m \in U\}.$$

Then  $S_{\eta_n} = \{m | d(m) \geq n\}$ .

## CHAPTER II

When considering commutative, weakly closed, symmetric subrings of  $B(H)$ , one often finds it fruitful to examine the following example. Let  $I$  denote the interval  $[0, 1]$ , and let  $\lambda$  denote ordinary Lebesgue measure on  $I$ . Let  $H$  denote the Hilbert space  $L_2(I, \lambda)$ . For each function  $\varphi \in L_\infty(I, \lambda)$ , we define an operator  $A_\varphi \in B(H)$  by  $A_\varphi f = \varphi f$  for all  $f \in L_2(I, \lambda)$ . Let  $R$  denote the ring of all such operators arising from  $L_\infty$  functions in this manner. Then  $R$  is a weakly closed, commutative, symmetric ring. It is shown in Theorem 2.2 that this example is somewhat more general than it might at first appear to be.

Definition 2.1 If  $R$  is a subring of  $B(H)$  and  $P$  is a projection in  $R$ , then  $P$  is said to be a minimal projection provided there is no projection  $Q \in R$  such that  $Q < P$ , and  $Q \neq 0$ .

In Theorem 2.2,  $I$  again denotes the interval  $[0, 1]$ ,  $\lambda$  denotes Lebesgue measure on  $I$ , and  $R$  denotes the ring  $\{A_\varphi | \varphi \in L_\infty(I, \lambda)\}$ .

Theorem 2.2 Suppose  $H$  is a separable Hilbert space and  $E$  is a maximal commutative, symmetric subring of  $B(H)$  which contains no minimal projections. Then there exists an isometry  $V$  from  $H$  onto  $L_2(I, \lambda)$  such that  $A \rightarrow VAV^{-1}$  is an

isometric isomorphism from  $E$  onto  $R$ .

Proof: Let  $M$  denote the maximal ideal space of  $E$ , and let  $\xi_0$  denote a unit cyclic vector for  $E$ . Let  $\mu$  be the measure on  $M$  defined by

$$\int_M A^{\wedge}(m) d\mu(m) = (A\xi_0 | \xi_0)$$

for all  $A \in E$ . Note that  $\mu(M) = 1$ .

Since  $H$  is separable, any subset of  $B(H)$  is separable in the strong operator topology. Therefore, there exist clopen subsets of  $M$ ,  $\{U_i\}_{i=1}^{\infty}$ , such that  $\{P_{U_i}\}_{i=1}^{\infty}$  is dense in the set of all projections in  $E$ . Let  $T_n$  be the smallest collection of clopen sets containing  $U_1, U_2, \dots, U_n$  such that the following are satisfied:

- i) if  $U, V \in T_n$  then  $U \cap V \in T_n$
- ii) if  $U \in T_n$  then  $M \setminus U \in T_n$ .

Let  $S_n = \{U \in T_n | U \neq \emptyset, \text{ and if } V \in T_n, V \subseteq U, \text{ then either } V = U \text{ or } V = \emptyset\}$ . Each  $S_n$  is a collection of pairwise disjoint clopen sets which comprises a partition of  $M$ . Furthermore, for each  $U \in S_{n+1}$ , there exists uniquely a clopen set  $U' \in S_n$  such that  $U \subseteq U'$ . Also,  $U_n$  is the union of some subcollection of  $S_n$ . Hence if  $\mathfrak{D} = \bigcup_n S_n$ , then  $\{P_U | U \in \mathfrak{D}\}$  is strongly dense in the set of all projections in  $E$ .

Claim: There exists a map  $T$  from  $\mathfrak{D}$  to a set of subintervals of  $[0, 1]$  such that  $T$  preserves Boolean operations and  $\mu(U) = \lambda(T(U))$  for each  $U \in \mathfrak{D}$ .

Proof of Claim: We can assume that  $U_1 \neq M$ . In that case,  $S_1 = \{U_1, M \setminus U_1\}$ . Let  $T(U_1) = [0, \mu(U_1)]$ , and let

$$T(M \setminus U_1) = [\mu(U_1), 1].$$

Suppose now that  $T$  has been defined on  $\mathfrak{D}_n = \bigcup_{i=1}^n S_i$  satisfying the conditions given in the claim. For each  $U \in S_n$ , we consider the set  $L_U = \{V \mid V \in S_{n+1} \setminus \mathfrak{D}_n, V \subset U\}$ . If  $L_U$  is non-empty, we enumerate its elements  $V_U^1, V_U^2, \dots, V_U^k$ . It is clear that we can partition  $T(U)$  with intervals  $I_1, I_2, \dots, I_k$  such that  $\mu(V_U^i) = \lambda(I_i)$ . Continuing this process for each  $U \in S_n$ , we define  $T$  on  $S_{n+1}$  and, therefore, by a maximality argument, on all of  $\mathfrak{D}$ .

For each  $m \in M$ , there exists a tower  $V_m^i$  of clopen sets such that  $m \in V_m^i$ , and  $V_m^i \in S_i$ . Suppose that

$$\lim_i \mu(V_m^i) = \varepsilon_0 > 0.$$

Then there exists a clopen set  $W$  such that  $\mu(W) = \varepsilon_0/2$  and  $W \subseteq \bigcap_i V_m^i$ . But for a given  $N$  and for all  $k > N$ ,  $U_N \cap V_m^k = V_m^k$  or  $\emptyset$ . Therefore

$$([P_{U_N} - P_W]\xi_0 | \xi_0) = \mu(U_N \setminus W) + \mu(W \setminus U_N) \geq \varepsilon_0/2,$$

which contradicts the denseness of the  $U_n$ 's. Hence,  $\mu(\bigcap_i V_m^i) = 0$ .

Now  $\lim_i \mu(V_m^i) = 0$  implies  $\lim_i \lambda(T(V_m^i)) = 0$ , and there exists a unique point  $x_m \in \bigcap_i T(V_m^i)$ . We define a function  $\zeta(m)$  by  $\zeta(m) = e^{2\pi i x_m}$ .  $\zeta(m)$  is clearly continuous, so there exists an operator  $U \in E$  such that  $\hat{U}(m) = \zeta(m)$ .



Since  $|U^\wedge(m)| = 1$  for all  $m \in M$ ,  $U$  is unitary.

Recall that  $S_k$  is a partition of  $M$ . For  $W \in S_k$ , let  $m(W)$  denote some element of  $W$ . Then

$$\sum_{W \in S_k} [U^\wedge(m(W))]^n \mu(W)$$

is an approximating sum for the integral

$$\int_M [U^\wedge(m)]^n d\mu(m).$$

Letting  $x_m$  be defined as before, we see

$$\sum_{W \in S_k} [U^\wedge(m(W))]^n \mu(W) = \sum_{W \in S_k} e^{2n\pi i x_{m(W)}} \lambda(T(W)).$$

But the sum on the right is an approximating sum for

$$\int_0^1 e^{2n\pi i \theta} d\theta.$$

Hence the powers of  $U^\wedge(m)$  are orthonormal in  $L_2(M, \mu)$ .

Since  $\{P_{U_1}\}_{i=1}$  is dense in the projections of  $E$  and since each  $P_{U_1}$  is a spectral projection of  $U$ , the powers - positive and negative - of  $U$  generate  $E$  in the weak operator topology. Equivalently, the powers of  $U^\wedge(m)$  generate  $L_\infty(M, \mu)$  in the weak topology. Hence we can define a map  $V_1$  from a dense subspace of  $L_2(M, \mu)$  to a dense subspace of  $L_2(I, \lambda)$  by

$$V_1 \sum_{p=1}^n a_p [U^\wedge(m)]^k p = \sum_{p=1}^n a_p e^{2\pi i k p \theta}.$$

$V_1$  is an isometric map which can be extended to an isometry from  $L_2(M, \mu)$  to  $L_2(I, \lambda)$ . Moreover,  $V$  is an

isometry in the  $L_\infty$  norm from  $L_\infty(M, \mu)$  to  $L_\infty(I, \lambda)$ . Because we can consider  $E$  to be the ring  $L_\infty(M, \mu)$  acting on  $L_2(M, \mu)$   $V$  possesses the desired properties.

At this point, certain observations concerning Theorem 2.2 are instructive. The operator  $U \in E$  defined in the proof of Theorem 2.2 is a bilateral shift on an orthonormal basis. Thus, we could have phrased our theorem to read that a ring satisfying the hypothesis of Theorem 2.2 is generated in the weak operator topology by a bilateral shift and its adjoint.

A second fact worth noting is that a shorter proof of Theorem 2.2 could be given using a general theorem on the structure of measure algebras, namely that a normalized, separable, non-atomic measure space is measure theoretically equivalent to  $[0, 1]$  under Lebesgue measure. The proof given for Theorem 2.2 essentially contains a proof of the theorem cited above. The proof given is the first one discovered by the author, and it makes the correspondence between  $E$  and  $R$  more explicit.

Theorem 2.2 not only specifies the structure of maximal, commutative, symmetric subrings of  $B(H)$  with no minimal projections; an easy corollary to Theorem 2.2 characterizes those maximal commutative symmetric rings which do have minimal projections. By the maximality of  $E$ , any minimal projection must project on a subspace of

dimension one. Therefore we can decompose  $H$  into a direct sum of an  $l_2$  space and an  $L_2$  space where  $E$  acts upon  $l_2$  via multiplication by bounded functions and  $E$  acts upon  $L_2$  by multiplication by  $L_\infty$  functions.

A natural question to ask is whether Theorem 2.2 can be expanded in some fashion to give a characterization of maximal commutative, symmetric subrings of  $B(H)$  for  $H$  non-separable. Our first insight to the problem is to note that the maximality in Theorem 2.2 was used solely to guarantee the existence of a cyclic vector. If  $H$  is separable, a weakly closed, commutative, symmetric subring of  $B(H)$  is maximal iff it has a cyclic vector. If  $H$  is not separable, however, such a ring need not have a cyclic vector, although a commutative, weakly closed, symmetric ring with a cyclic vector is necessarily maximal.

If we assume that  $E$  is a weakly closed, commutative, symmetric ring with a cyclic vector, then we can consider  $E$  to be an  $L_\infty$  ring acting via multiplication on an  $L_2$  space. The desired extension of Theorem 2.2 follows easily from a theorem by D. Maharam Stone on the structure of measure algebras. If  $\alpha$  is a cardinal number, let  $T_\alpha$  denote the topological product of  $\alpha$  copies of  $[0, 1]$ . Let  $\mu_\alpha$  denote the product Lebesgue measure on  $T_\alpha$ , and let  $H_\alpha = L_2(T_\alpha, \mu_\alpha)$ . Suppose  $(M, \mu)$  is a measure space such that  $\dim L_2(M, \mu) = \alpha$ . If  $K$  is a measurable subset of  $M$ , then  $\pi_K$  gives rise to the projection  $Q^K \in B(L_2(M, \mu))$

defined by  $Q^K f = \pi_K f$ . If the subspace on which  $Q^K$  projects has dimension  $\alpha$  for every set  $K$  of positive measure, then  $(M, \mu)$  is said to be homogeneous. The Maharam Stone theorem to which we alluded above says that a normalized homogeneous measure space  $(M, \mu)$  such that  $\dim L_2(M, \mu) = \alpha$  is measure theoretically equivalent to  $(T_\alpha, \mu_\alpha)$ . That is to say, there exists a one to one measure preserving transformation from the equivalence classes of measurable subsets of  $M$  to the equivalence classes of measurable subsets of  $T_\alpha$  which preserves the Boolean operations. This transformation induces in an obvious manner an isometry from measurable step functions of  $L_2(M, \mu)$  to measurable step functions in  $L_2(T_\alpha, \mu_\alpha)$  which can be extended to an isometry  $V$  from  $L_2(M, \mu)$  to  $L_2(T_\alpha, \mu_\alpha)$ . But  $V$  is also an isometry in  $L_\infty$  norm. Thus we arrive at the following theorem.

Theorem 2.3 Suppose that  $H$  is a Hilbert space of dimension  $\alpha$  and  $E$  is a commutative, symmetric, weakly closed subring of  $B(H)$  with a cyclic vector. Furthermore, suppose that each projection in  $E$  projects on a subspace of dimension  $\alpha$ . Then there exists an isometry  $V$  from  $H$  to  $L_2(T_\alpha, \mu_\alpha)$  such that  $A \rightarrow VAV^{-1}$  is an isometric isomorphism from  $E$  to  $L_\infty(T_\alpha, \mu_\alpha)$ .

In general, if  $E$  is a weakly closed, commutative,

symmetric ring with a cyclic vector and containing no minimal projections, then  $H$  can be decomposed into a direct sum

$$H = \sum_{\alpha \in \mathfrak{D}} L_2(T_\alpha, \mu_\alpha)$$

where  $\mathfrak{D}$  is some denumerable collection of cardinals, and  $E$  is completely isomorphic to the complete direct sum of the rings  $L(T_\alpha, \mu_\alpha)$  acting on  $L_2(T_\alpha, \mu_\alpha)$ .

We now wish to extend Theorem 2.2 in a different direction, and we motivate our considerations with an example.

Example 2.4 We again use  $(I, \lambda)$  to denote the measure space given by the usual measure on  $[0, 1]$ . Let  $L$  denote the Hilbert space  $L_2(I, \lambda)$ , and for each  $\varepsilon$  such that  $0 < \varepsilon < 1$ , let  $L_\varepsilon$  denote the set of all functions in  $L$  which vanish almost everywhere on the interval  $(\varepsilon, 1)$ . Let  $\pi_\varepsilon = \pi_{(0, \varepsilon)}$ . For some fixed  $\varepsilon$ , we consider the space  $H = L + L_\varepsilon$ . For each  $\varphi \in L(I, \lambda)$ , define  $A_\varphi \in B(H)$  by  $A_\varphi(f, g) = (\varphi f, \varphi g)$ , and denote by  $E$  the ring of all operators obtained in this manner. If  $\zeta, \eta \in L(I, \lambda)$ , we can define an operator  $B_{\zeta, \eta} \in B(H)$  by

$$B_{\zeta, \eta}(f, g) = (\zeta f, \eta g). \text{ We also define}$$

an operator  $U \in B(H)$  by  $U(f, g) = (g, \pi_\varepsilon f)$ . It is trivial to verify that  $U \in E'$  and that for each choice of  $\zeta$  and  $\eta$ ,  $B_{\zeta, \eta} \in E'$ .  $U$ , however, does not, in general, commute with the operators  $B_{\zeta, \eta}$ .

Claim:  $\{U\} \cup \{B_{\zeta,\eta} \mid \zeta,\eta \in L_\infty(I, \lambda)\}$  generates  $E'$ .

Proof of claim: Suppose  $UA = AU$  and  $AB_{\zeta,\eta} = B_{\zeta,\eta}A$  for all  $\zeta,\eta \in L_\infty(I, \lambda)$ . We need only show that  $A \in E$ . Suppose  $A(1,0) = (\xi_1, \xi_2)$ . Then

$$\begin{aligned} (\xi_1, \xi_2) &= A(1,0) = AB_{1,0}(1,0) \\ &= B_{1,0}A(1,0) = B_{1,0}(\xi_1, \xi_2) \\ &= (\xi_1, 0) \end{aligned}$$

Therefore  $\xi_2 = 0$ . Also,

$$A(0, \pi_\varepsilon) = AU(1,0) = UA(1,0) = U(\xi_1, 0) = (0, \pi_\varepsilon \xi_1)$$

Suppose  $\zeta, \eta \in L_\infty(I, \lambda)$ . Then

$$\begin{aligned} A(\zeta, \pi_\varepsilon \eta) &= A(\zeta, 0) + A(0, \pi_\varepsilon \eta) \\ &= AB_{\zeta,0}(1,0) + AB_{0,\eta}(0, \pi_\varepsilon) \\ &= B_{\zeta,0}A(1,0) + B_{0,\eta}A(0, \pi_\varepsilon) \\ &= (\xi_1 \zeta, 0) + (0, \eta \pi_\varepsilon \xi_1) \\ &= (\xi_1 \zeta, \xi_1 \eta \pi_\varepsilon) = A_{\xi_1}(\zeta, \pi_\varepsilon \eta). \end{aligned}$$

Since  $A$  agrees with  $A_{\xi_1}$  on a dense set,  $A = A_{\xi_1}$ , and  $A \in E$ .

Let  $M_1$  denote the set of all vectors in  $H$  of the form  $(f, 0)$ ,  $M_2$  the set of all vectors in  $H$  of the form  $(0, g)$ , and  $K$  the set of all vectors  $(f, g) \in H$  such that  $\pi_\varepsilon f = f$ . Then  $E$  applied to  $(1, 0)$  generates  $M_1$ , and  $E$  applied to  $(0, \pi_\varepsilon)$  generates  $M_2$ .  $(1, 0)$  is cyclic for  $E'$ , and  $(0, \pi_\varepsilon)$  is cyclic in  $K$  for  $E'$ . Therefore,

$$\{(M_1, (1, 0)), (M_2, (0, \pi_\varepsilon))\}$$

is a canonical decomposition system for  $E$ .

Our next theorem shows that the ring in Example 2.4 is typical of a large class of rings.

Theorem 2.5 Suppose  $H$  is a separable Hilbert space and  $E$  is a weakly closed, commutative, symmetric subring of  $B(H)$  which contains the identity but contains no minimal projections. Let  $M$  denote the maximal ideal space of  $E$ ,  $\xi_0$  a cyclic vector for  $E$ , and  $\mu$  the measure on  $M$  arising from  $\xi_0$ . Let  $\{(K_i, \eta_i)\}$  be a canonical decomposition system for  $E$ , and let  $\varepsilon_i = \mu(S_{\eta_i})$ . Defining  $L_{\varepsilon}$  as before, we let  $L = L_{\varepsilon_1} + L_{\varepsilon_2} + \dots$ . For  $\varphi \in L(I, \lambda)$ , define  $A_\varphi \in B(L)$  by  $A_\varphi(f_1, f_2, \dots) = (\varphi f_1, \varphi f_2, \dots)$ . We let  $R$  denote the ring  $\{A_\varphi | \varphi \in L(I, \lambda)\}$ . Then there exists an isometry  $V$  from  $H$  onto  $L$  such that  $A \mapsto VAV^{-1}$  maps  $E$  to  $R$ , and  $V(K_i) = L_{\varepsilon_i}$ .

Proof: We select a sequence  $\{i_n\}$  in the following manner. Let  $i_1 = 1$ . Let  $i_2$  denote the least value of  $k$  for which  $\varepsilon_k < 1$ . Let  $i_3$  denote the least value of  $k$  for which  $\varepsilon_k < \varepsilon_{i_2}$ . Continuing inductively, we obtain a sequence  $\{i_n\}$  such that  $\varepsilon_{i_{n+1}} < \varepsilon_{i_n}$  and if  $i_n \leq k < i_{n+1}$ , then  $\varepsilon_k = \varepsilon_{i_n}$ .

If  $T \subseteq M$ , we let  $L_2(T, M, \mu)$  denote the set of functions in  $L_2(M, \mu)$  which vanish almost everywhere off  $T$ . Then we may consider  $K_i$  to be  $L_2(S_{\eta_i}, M, \mu)$ , where  $E$  acts on  $K_i$  by

multiplication by  $L_\infty$  functions. Let  $\lambda$  denote Lebesgue measure on  $I = [0,1]$ . For  $S \subseteq I$ , let  $L_2(S, I, \lambda)$  denote the functions of  $L_2(I, \lambda)$  which vanish almost everywhere off  $S$ . By the proof of Theorem 2.2, there exists an isometry  $U_n$  from  $L_2(S_{\eta_{i_n}} \setminus S_{\eta_{i_{n+1}}}, M, \mu)$  onto  $L_2([\varepsilon_{i_{n+1}}, \varepsilon_{i_n}], I, \lambda)$  with the property that for each  $\phi \in L_\infty(M, \mu)$  with support in  $S_{\eta_{i_n}} \setminus S_{\eta_{i_{n+1}}}$ ,  $\|U_n \phi\| = \|\phi\|$ . To define our isometry  $V$  from  $H$  onto  $L$ , we consider two cases.

Case I:  $\lim_n \varepsilon_{i_n} = 0$ .

Suppose  $f \in K_t$  - which we identify with  $L_2(S_{\eta_t}, M, \mu)$  - where  $i_n \leq t < i_{n+1}$ . Then  $f$  can be written uniquely as a sum

$$f = \sum_{p \geq n} f_p$$

where  $f_p \in L_2(S_{\eta_{i_p}} \setminus S_{\eta_{i_{p+1}}}, M, \mu)$ . We define  $Vf \in L_{\varepsilon_t}$  by

$$Vf = \sum_{p \geq n} U_p f_p.$$

Extending  $V$  to all of  $H$  in the obvious manner clearly yields an isometry, and  $A \rightarrow VAV^{-1}$  is an isometric isomorphism from  $E$  to  $R$ .

Case II:  $\lim_n \varepsilon_{i_n} = \alpha_0 > 0$ .

In this case we define an isometry  $U_\infty$  as follows:

Let  $S_\infty = M \setminus \overline{\bigcup_1 (M \setminus S_{\eta_1})}$ . Then  $S_\infty$  is clopen, and



$\mu(S_\infty) = \alpha_0$ . We let  $U_\infty$  be a map from  $L_2(S_\infty, M, \mu)$  onto  $L_2([0, \alpha_0], I, \lambda)$  obtained in the same manner as before which is an isometry in both the  $L_2$  and  $L_\infty$  norms. Then for  $f \in K_t$ ,  $i_n \leq t < i_{n+1}$  we can write

$$f = \sum_{p \geq n} f_p + f_\infty$$

where  $f_p \in L_2(S_{\eta_{i_p}} \setminus S_{\eta_{i_{p+1}}}, M, \mu)$  and  $f_\infty \in L_2(S_\infty, M, \mu)$

Defining  $Vf \in L_{\varepsilon_t}$  by

$$Vf = \sum_{p \geq n} U_p f_p + U_\infty f_\infty$$

yields an isometry with the desired properties.

We note that the numbers  $\varepsilon_i$  play only a marginal role in Theorem 2.5 since they are not uniquely determined by the ring  $E$ , but depend also on the cyclic vector  $\xi_0$ .  $E$  only fixes the sequence  $\{i_n\}$  and dictates whether or not  $\lim_n \varepsilon_{i_n} = 0$ .

### CHAPTER III

Our goal in this chapter is to develop sets of necessary and sufficient conditions for two normal operators on a separable Hilbert space to be unitarily equivalent. We shall do this by exploiting the structure of rings of operators, in particular, weakly closed, commutative, symmetric rings and the associated maximal ideal space theory. As one might suspect, we shall be led to a theory of spectral multiplicity, but more of that presently.

What is desired is a set of properties (or, more precisely, functions) which (within unitary equivalence) completely determine a normal operator. In the quest of some such set of properties, a first naive guess might be that a normal operator is geometrically characterized by its spectrum and the symmetric ring that the operator generates in the weak (or perhaps norm) topology. Example 3.4 justifies the use of the word naive above and suggests a way in which the guess might be modified.

Definition 3.1 Let  $S$  denote the Cantor ternary set and  $S^C$  the complement of  $S$  relative to  $[0,1]$ . For each positive integer  $n$ , let  $E_n^1, E_n^2, \dots, E_n^{2^{n-1}}$  denote the components of  $S^C$  of length  $(1/3)^n$  indexed in such a way that  $E_n^i$  lies to the left of  $E_n^j$  whenever  $1 \leq i < j \leq 2^{n-1}$ .

For  $x \in E_n^k$ , let  $C_1(x) = (2k-1)/2^n$ . The Cantor function, which we denote by  $C(x)$ , is the unique continuous extension to  $[0,1]$  of  $C_1(x)$ .

For our purposes, the pertinent properties of  $C(x)$  are the following:

- i)  $C(S) = [0,1]$ ;
- ii)  $C(x)$  is monotone increasing;
- iii)  $C(x)$  is continuous.

These properties are proved in [1] p. 238-239.

Consider the Hilbert space  $H = L_2([0,1])$  where the measure is ordinary Lebesgue measure. Associated with each function  $\alpha(x) \in L([0,1])$  is a normal operator  $A_\alpha$  defined by  $(A_\alpha f)(x) = \alpha(x)f(x)$  for all  $f \in L_2([0,1])$ . The ring of operators (which we shall call  $L$  -- making no distinction between the set of operators and the set of functions) generated in this manner is a maximal commutative symmetric subring of  $B(H)$ .

Lemma 3.2 If  $f(x)$  is continuous on  $[0,1]$ , then the spectrum of  $A_f$  is the range of  $f$ .

Proof: Since  $L$  is a weakly closed, symmetric ring, the spectrum of  $A_f$  with respect to the ring  $L$  is the same as the spectrum of  $A_f$  in  $B(H)$ . Hence  $\lambda$  is in the spectrum of  $A_f$  iff  $[f(x) - \lambda]^{-1} \in L$ . But  $[f(x) - \lambda]^{-1} \in L$  iff  $\lambda$  is not in the range of  $f$ .

Lemma 3.3 Suppose  $f(x)$  is a continuous, real-valued, strictly increasing function on  $[0,1]$ . Let  $E_\lambda = \{x | f(x) \leq \lambda\}$ , and let  $g_\lambda(x) = \pi_{E_\lambda}(x)$ . If  $P(\lambda)$  denotes the spectral function of  $A_f$ , then  $P(\lambda) = A_{g_\lambda}$ .

Proof: Let  $Q(\lambda) = A_{g_\lambda}$ . The range of  $f(x)$  is some closed interval  $[a,b]$ . Suppose  $\varepsilon > 0$  and consider a partition of  $[a,b]$   $\{a = \lambda_0 < \lambda_1 < \dots < \lambda_n = b\}$  with the maximum of  $\{\lambda_i - \lambda_{i-1}\}_{i=1}^n < \varepsilon$ . For  $i = 1, 2, \dots, n$ , let

$$\lambda'_i \in [\lambda_{i-1}, \lambda_i].$$

If  $x \in E_{\lambda_k} \setminus E_{\lambda_{k-1}}$ , then

$$|f(x) - \sum_{i=1}^n \lambda'_i [g_{\lambda_i}(x) - g_{\lambda_{i-1}}(x)]| = |f(x) - \lambda'_k| < \varepsilon.$$

Since the inequality holds for all but perhaps a finite number of values of  $x$ ,

$$||A_f - \sum_{i=1}^n \lambda'_i [Q(\lambda_i) - Q(\lambda_{i-1})]| < \varepsilon.$$

Therefore,  $Q(\lambda) = P(\lambda)$ .

We have now laid the necessary groundwork to give a simple presentation of our example.

Example 3.4 Let  $f(x) = x + C(x)$  and  $g(x) = 2x$ . Then  $A_f$  and  $A_g$  have the same spectra,  $A_f$  and  $A_g$  generate the same symmetric rings in the norm topology,  $A_f$  and  $A_g$  generate the same symmetric rings in the weak topology, but  $A_f$  is not unitarily equivalent to  $A_g$ .

Proof: By Stone's Theorem,  $A_f$  and  $A_g$  (each) generate in the norm topology the ring of all operators arising from continuous functions which vanish at 0. This ring is weakly dense in  $L_\infty$ ; hence, each operator generates  $L_\infty$  in the weak topology. Since  $f(x)$  and  $g(x)$  are continuous and have range  $[0,2]$ , by Lemma 3.2,  $A_f$  and  $A_g$  have the same spectra.

Let  $P(\lambda)$  and  $Q(\lambda)$  denote the spectral functions of  $A_f$  and  $A_g$  respectively. Let  $\mathfrak{D} = f(S)$ , where  $S$  again denotes the Cantor ternary set. If there exists a unitary operator  $U$  such that  $UA_fU^{-1} = A_g$ , then  $UP(\mathfrak{D})U^{-1} = Q(\mathfrak{D})$ . Because  $P(\mathfrak{D})$  is multiplication by the characteristic function of  $S$ ,  $P(\mathfrak{D}) \neq 0$ . Since  $\mathfrak{D}$  has measure 1,  $\{x | g(x) \in \mathfrak{D}\}$  has measure 1/2. Therefore,  $Q(\mathfrak{D})$  is multiplication by the characteristic function of a set of measure 1/2, and  $Q(\mathfrak{D}) \neq 0$ . This contradicts our assumption that  $A_f$  and  $A_g$  are unitarily equivalent.

The basic difference between  $A_f$  and  $A_g$  in the above example seems to be that  $f(x)$  takes a set of zero measure to a set of positive measure whereas  $g(x)$  does not. When one considers that each maximal commutative, symmetric subring of  $B(H)$  for  $H$  separable can be realized as an  $L_\infty$  ring, the example hints at what our necessary and sufficient conditions might be.

In Theorems 3.5 - 3.12  $N_1$  and  $N_2$  denote normal operators in  $B(H)$ , where  $H$  is a separable Hilbert space.  $E^1$

denotes the symmetric ring generated by  $N_1$  in the weak topology, and  $M_1$  denotes the maximal ideal space of  $E^1$ .  $A \rightarrow A^\wedge(m)$  denotes the Gelfand transform from  $E^1$  to  $C(M_1)$ .

Theorem 3.5 Suppose  $U$  is a unitary operator such that  $UN_1 = N_2U$ . If  $\mathfrak{D}$  is a measurable subset of the complex plane, then  $N_1^{\wedge-1}(\mathfrak{D})$  has void interior iff  $N_2^{\wedge-1}(\mathfrak{D})$  has void interior.

Proof: The correspondence  $A \rightarrow UAU^{-1}$  is an isometric isomorphism from  $E^1$  to  $E^2$ . It induces a homeomorphism  $\Phi: M_1 \rightarrow M_2$ , and this homeomorphism has the property that for each complex number  $z$ ,  $N_1^{\wedge}(m) = z$  iff  $N_2^{\wedge}(\Phi(m)) = z$ .

Theorem 3.6 Suppose that  $E^1$  and  $E^2$  are maximal commutative and that for each measurable set  $\mathfrak{D}$  of complex numbers,  $N_1^{\wedge-1}(\mathfrak{D})$  has void interior iff  $N_2^{\wedge-1}(\mathfrak{D})$  has void interior. Then  $N_1$  is unitarily equivalent to  $N_2$ .

Proof: Our first step in the proof of Theorem 3.6 is to prove the following lemma.

Lemma 3.7 Suppose  $E$  is a maximal commutative, symmetric subring of  $B(H)$  generated in the weak topology by  $N$  and  $N^*$ . Let  $M$  denote the maximal ideal space of  $E$ , and let  $U$  be a clopen subset of  $M$ . Then there exists a measurable set  $\mathfrak{D}$  of complex numbers such that the symmetric difference of  $U$  and  $N^{\wedge-1}(\mathfrak{D})$  has void interior.

Proof: If  $F$  and  $G$  are measurable subsets of  $M$ , we shall write  $F \simeq G$  to mean that the symmetric difference of  $F$  and  $G$  has void interior. Let  $\mathfrak{D}_1 = N^\wedge(U)$ . Then there exists a clopen set  $V$  such that  $V \simeq N^{\wedge^{-1}}(\mathfrak{D}_1)$ . Note that  $U \subset V$ . Let  $\xi_0$  be a unit cyclic vector for  $E$ , and let  $\mu$  denote the corresponding measure on  $M$ . Since  $N$  and  $N^*$  generate  $E$  in the weak topology, there exist polynomials  $p_n$  in  $N$  and  $N^*$  such that

$$\int_M |p_n^\wedge(m) - \pi_U(m)| d\mu(m) < 1/2^n.$$

Let  $S_n = \{m \in V \mid |p_n^\wedge(m)| \leq 1/2\}$  and

$$T_n = \{m \in V \mid |p_n^\wedge(m)| > 1/2\}.$$

Then  $S_n \cup T_n = V$ , and  $S_n \cap T_n = \emptyset$  for every  $n$ . Let  $\varepsilon_2 = \mu(V)$  and  $\varepsilon_1 = \mu(U)$ . We may assume that  $\varepsilon_2 > \varepsilon_1 > 0$ . For  $m \in S_n \cap U$ ,  $|p_n^\wedge(m) - \pi_U(m)| \geq 1/2$ , and

$$\begin{aligned} 1/2^n &> \int_M |p_n^\wedge(m) - \pi_U(m)| d\mu(m) \\ &> \int_{S_n \cap U} |p_n^\wedge(m) - \pi_U(m)| d\mu(m) \\ &\geq (1/2)\mu(S_n \cap U). \end{aligned}$$

Hence  $\mu(S_n \cap U) < 1/2^{n-1}$  and  $\mu(T_n \cap U) > \varepsilon_1 - 1/2^{n-1}$ .

For  $m \in T_n \cap (V \setminus U)$ ,  $|p_n^\wedge(m) - \pi_U(m)| > 1/2$ , and

$$\begin{aligned} 1/2^n &> \int_M |p_n^\wedge(m) - \pi_U(m)| d\mu(m) \\ &\geq \int_{T_n \cap (V \setminus U)} |p_n^\wedge(m) - \pi_U(m)| d\mu(m) \end{aligned}$$

$$\geq (1/2)\mu(T_n \cap [V \setminus U]).$$

Therefore  $\mu(T_n \cap (V \setminus U)) < 1/2^{n-1}$ , and

$$\mu(S_n \cap (V \setminus U)) > \varepsilon_2 - \varepsilon_1 - 1/2^{n-1}.$$

Let  $K = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} S_i$  and let  $L = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} T_i$ .  $K$  and  $L$  are measurable subsets of  $V$ .  $K$  and  $L$  are disjoint since  $m_1 \in K \cap L$  implies that for sufficiently large  $n$ ,  $m_1 \in S_n \cap T_n$ . But  $S_n \cap T_n = \emptyset$  for all  $n$ .

$$\begin{aligned} \text{Now } \mu\left(\bigcup_{n=N}^{\infty} T_n\right) &= \mu\left(\bigcup_{n=N}^{\infty} [U \setminus T_n]\right) = \mu\left(\bigcup_{n=N}^{\infty} [U \cap S_n]\right) \\ &\leq \sum_{n=N}^{\infty} \mu(U \cap S_n) < \sum_{n=N}^{\infty} 1/2^n = 1/2^{N-1}. \end{aligned}$$

Hence  $\mu(U \setminus L) = 0$ . Also,  $L \setminus U \subset \bigcap_{n \geq N} T_n \setminus U \subset T_n \setminus U$   
 $\subset T_n \cap [V \setminus U].$

But  $\mu(T_n \cap [V \setminus U]) < \frac{1}{2^{N-1}}$ . Therefore,  $\mu(L \setminus U) = 0$ . Similarly,  $\mu([V \setminus U] \setminus K) = \mu(K \setminus [V \setminus U]) = 0$ . But this implies that  $U \simeq L$  and  $(V \setminus U) \simeq K$ . Let  $\mathfrak{D} = N(L)$ . If  $m \in K$ , then for sufficiently large  $n$ ,  $|p_n^{\wedge}(m)| \leq 1/2$ . Likewise, for  $m \in L$ ,  $|p_n^{\wedge}(m)| > 1/2$  if  $n$  is sufficiently large. Thus, if  $m_1 \in L$  and  $m_2 \in K$ , then  $N^{\wedge}(m_1) \neq N^{\wedge}(m_2)$ . Therefore,  $N^{\wedge^{-1}}(\mathfrak{D}) \cap K = \emptyset$ , and we see that  $U \simeq N^{\wedge^{-1}}(\mathfrak{D})$ .

We are now prepared to proceed with the proof of Theorem 3.6. Let  $\xi_i$  be a unit cyclic vector for  $E^i$ , and let  $\mu_i$  be the corresponding measure on  $M_i$ . Suppose that  $\mathfrak{D}_1$  and  $\mathfrak{D}_2$  are measurable sets of complex numbers such that



$N_2^{-1}(\mathfrak{D}_1) \simeq N_2^{-1}(\mathfrak{D}_2)$ . Then  $N_1^{-1}(\mathfrak{D}_1) \simeq N_1^{-1}(\mathfrak{D}_2)$ , and consequently,  $\mu_1[N_1^{-1}(\mathfrak{D}_1)] = \mu_1[N_1^{-1}(\mathfrak{D}_2)]$ . Therefore we can define a measure  $\alpha$  on  $M_2$  in the following manner: For each clopen subset  $U$  of  $M_2$  there exists a measurable set of complex numbers  $\mathfrak{D}$  such that  $U \simeq N_2^{-1}(\mathfrak{D})$ . Let  $\alpha(U) = \mu_1(N_1^{-1}(\mathfrak{D}))$ . If  $K$  is a subset of  $M_2$  measurable with respect to  $\mu_2$ , then  $K \simeq V$  for some clopen subset  $V$  of  $M_2$ . Let  $\alpha(K) = \alpha(V)$ .  $\alpha$  is a positive measure supported on all of  $M_2$ ,  $\alpha(M_2) = 1$ , and  $\alpha$  is absolutely continuous with respect to  $\mu_2$ . Consequently, there exists a function  $\phi(m) \in L_1(M_2, \mu_2)$  such that

$$\int_{M_2} \hat{A}(m) d\alpha(m) = \int_{M_2} \hat{A}(m)\phi(m) d\mu_2(m)$$

for all  $A \in E^2$ . Note that  $\phi(m) \geq 0$ . There exists a sequence  $\{A_n\}_{n=1}$  of operators in  $E^2$  such that

$0 \leq \hat{A}_n(m) \leq \hat{A}_{n+1}(m)$  and  $\|\hat{A}_n(m) - \phi(m)\|_1 \rightarrow 0$ . Let  $B_n$  denote the unique operator in  $E^2$  such that  $B_n^{\wedge}(m) \geq 0$  and  $B_n^2 = A_n$ .

Claim:  $B_n \xi_2$  converges.

Proof:  $\hat{A}_n(m)$  is Cauchy in  $L_1$  norm. Hence

$$\int_{M_2} |\hat{A}_i(m) - \hat{A}_j(m)| d\mu_2(m) \rightarrow 0.$$

Suppose  $i > j$ . Then

$$|B_i \xi_2 - B_j \xi_2|^2 = ([B_i - B_j] \xi_2 | [B_i - B_j] \xi_2)$$

$$\begin{aligned}
& \leq |(B_1 \xi_2 | \xi_2) - (B_1 B_j \xi_2 | \xi_2)| + |(B_1 B_j \xi_2 | \xi_2) - (B_j^2 \xi_2 | \xi_2)| \\
& = \int_{M_2} [A_i^{\wedge}(m) - B_i^{\wedge}(m) B_j^{\wedge}(m)] d\mu_2(m) \\
& \quad + \int_{M_2} [B_i^{\wedge}(m) B_j^{\wedge}(m) - A_j^{\wedge}(m)] d\mu_2(m) \\
& \leq 2 \int_{M_2} [A_i^{\wedge}(m) - A_j^{\wedge}(m)] d\mu_2(m)
\end{aligned}$$

which is near zero for large values of  $i$  and  $j$ . Therefore  $B_n \xi_2$  is Cauchy and converges to some  $\xi'_2 \in H$ .

$$\begin{aligned}
\text{Now for } A \in E^2, \quad \int_{M_2} A^{\wedge}(m) d\alpha(m) &= \int_{M_2} A^{\wedge}(m) \phi(m) d\mu_2(m) \\
&= \lim_n \int_{M_2} A^{\wedge}(m) A_n^{\wedge}(m) d\mu_2(m) \\
&= \lim_n (A A_n \xi_2 | \xi_2) \\
&= \lim_n (A B_n \xi_2 | B_n \xi_2) = (A \xi'_2 | \xi'_2).
\end{aligned}$$

Since  $\alpha(M_2) = 1$ ,  $|\xi'_2| = 1$ . Also,  $\xi'_2$  is cyclic for  $E^2$ . If it were not, then  $S_{\xi'_2} \neq M_2$  and  $\alpha(M_2 \setminus S_{\xi'_2}) = 0$ . But for some measurable set  $\mathfrak{D}$ ,  $M_2 \setminus S_{\xi'_2} \simeq N_2^{-1}(\mathfrak{D})$ , and  $N_1^{-1}(\mathfrak{D})$  has nonvoid interior. But then  $0 < \mu_1[N_1^{-1}(\mathfrak{D})] = \alpha(M_2 \setminus S_{\xi'_2})$  which is a contradiction.

We define a function  $\theta: C(M_1) \rightarrow C(M_2)$  as follows. If  $U_1, U_2, \dots, U_n$  is a partition of  $M_1$  where each  $U_i$  is clopen, then there exist measurable sets of complex numbers  $\mathfrak{D}_1, \mathfrak{D}_2, \dots, \mathfrak{D}_n$  such that  $N_1^{-1}(\mathfrak{D}_i) \simeq U_i$  for each  $i$ . There exist clopen subsets  $V_1, V_2, \dots, V_n$  of  $M_2$  such that  $V_i \simeq N_2^{-1}(\mathfrak{D}_i)$

for each  $i$ .  $\{V_i\}_{i=1}^n$  is a partition of  $M_2$ . If  $f(m) =$

$$\sum_{i=1}^n a_i \pi_{U_i}(m), \text{ then we let}$$

$$\Theta(f(m)) = \sum_{i=1}^n a_i \pi_{V_i}(m). \text{ Our assumption that}$$

$$N_1^{-1}(\mathfrak{D}) \simeq \emptyset \text{ iff } N_2^{-1}(\mathfrak{D}) \simeq \emptyset$$

guarantees that  $\Theta$  is well defined.  $\Theta$  is an isometry in  $L_\infty$  norm from a set dense in  $L_\infty(M_1, \mu_1)$  to a set dense in  $L_\infty(M_2, \alpha)$ . Therefore  $\Theta$  can be extended to an isometry from  $L_\infty(M_1, \mu_1)$  to  $L_\infty(M_2, \alpha)$  or, equivalently, from  $C(M_1)$  to  $C(M_2)$ . We may consider  $\Theta$  as a function from  $E^1$  to  $E^2$ . If  $P(\mathfrak{D})$  denotes the spectral function for  $N_1$  and  $Q(\mathfrak{D})$  denotes the spectral function for  $N_2$ , then  $\Theta[P(\mathfrak{D})] = Q(\mathfrak{D})$ . Hence  $\Theta(N_1) = N_2$ , and  $\Theta(N_1^*) = N_2^*$ . Therefore  $\Theta$  is a symmetric isomorphism.  $\Theta(AB) = \Theta(A)\Theta(B)$  whenever  $A^\wedge(m)$  and  $B^\wedge(m)$  are clopen step functions on  $M_1$  or when  $A$  and  $B$  are polynomials in  $N_1$  and  $N_1^*$ . Either fact together with the continuity of  $\Theta$  in the operator norm implies that  $\Theta(AB) = \Theta(A)\Theta(B)$  for all  $A, B \in E^1$ .

The final property of  $\Theta$  that we shall need to utilize is that  $\Theta$  is an isometry in  $L_1$  norm, i.e.

$$\int_{M_1} A^\wedge(m) d\mu_1(m) = \int_{M_2} \Theta(A)^\wedge(m) d\alpha(m)$$

for all  $A \in E^1$ . This is true because the  $L_1$  norm is less than or equal to the  $L_\infty$  norm and

$$\int_{M_1} A^\wedge(m) d\mu_1(m) = \int_{M_2} \Theta(A)^\wedge(m) d\alpha(m)$$

whenever  $\hat{A}(m)$  is a clopen step function.

We now define an operator  $U \in B(H)$  by  $UA\xi_1 = \theta(A)\xi_2'$  for each  $A \in E^1$ . Since  $|UA\xi_1|^2 = (UA\xi_1 | UA\xi_1)$

$$\begin{aligned} &= (\theta(A)\xi_2' | \theta(A)\xi_2') = \int_{M_2} \theta(A^*A)\hat{A}(m) d\alpha(m) \\ &= \int_{M_1} (A^*A)\hat{A}(m) d\mu_1(m) = |A\xi_1|^2, \end{aligned}$$

$U$  can be extended to a unitary operator.

For  $A \in E^1$ ,  $UN_1A\xi_1 = \theta(N_1A)\xi_2' = \theta(N_1)\theta(A)\xi_2' = N_2UA\xi_1$ . Therefore,  $UN_1 = N_2U$  on a dense subset of  $H$ , from which it follows that  $UN_1 = N_2U$  on all of  $H$ .

Example 3.8: Let  $f(x) = x$  and  $g(x) = x^2$  be defined on  $[0, 1]$ . As an application of Theorem 3.6, we shall show that  $A_f$  and  $A_g$  considered as operators on  $L_2([0, 1])$  are unitarily equivalent.

$A_f$  and  $A_g$  each generate  $L_\infty$  in the weak operator topology. Let  $P(\mathfrak{D})$  and  $Q(\mathfrak{D})$  denote the spectral functions of  $A_f$  and  $A_g$  respectively. If the hypothesis of Theorem 3.6 is not satisfied, then there exists a measurable set  $\mathfrak{D}$  such that either  $P(\mathfrak{D}) = 0$  and  $Q(\mathfrak{D}) \neq 0$  or  $P(\mathfrak{D}) \neq 0$  and  $Q(\mathfrak{D}) = 0$ . But by Lemma 3.3, it is clear that there exists no such  $\mathfrak{D}$ . Hence, by Theorem 3.6,  $A_f$  is unitarily equivalent to  $A_g$ .

We now wish to consider the general case; that is, we wish to drop the assumption that  $E^1$  and  $E^2$  are maximal. However, we shall assume henceforth that  $E^1$  and  $E^2$  contain

the identity operator. This is actually not a serious restriction, for if  $P_1$  is the principle identity of  $E^1$ , then our theorems will give conditions under which  $N_1|_{P_1(H)}$  is unitarily equivalent to  $N_2|_{P_2(H)}$ . If  $N_1$  restricted to  $P_1(H)$  is unitarily equivalent to  $N_2$  restricted to  $P_2(H)$ , then  $N_1$  is unitarily equivalent to  $N_2$  iff

$$\dim P_1(H)^\perp = \dim P_2(H)^\perp.$$

We can define an equivalence relation on the measurable subsets of the complex numbers by  $\mathfrak{D}_1 \simeq \mathfrak{D}_2$  if  $N_1^{-1}(\mathfrak{D}_1) \simeq N_1^{-1}(\mathfrak{D}_2)$ . The induced equivalence classes will be called the spectral classes of  $N_1$ . Suppose  $\Omega$  is a spectral class of  $N_1$  and  $\mathfrak{D} \in \Omega$ . Let  $\{(K_1, \eta_j)\}$  be a canonical decomposition for  $E^1$ . If  $n$  is the least positive integer for which it is true that  $N_1^{-1}(\mathfrak{D}) \cap S_{\eta_{n+1}}$  has void interior, then we will say that  $\Omega$  has multiplicity  $n$ . If  $N_1^{-1}(\mathfrak{D}) \cap S_{\eta_{n+1}}$  has non-void interior for all  $n$ , then we will say that  $\Omega$  has infinite multiplicity. Recall that the  $S_{\eta_j}$ 's are uniquely determined by the ring  $E^1$ . Hence there is no ambiguity in our definition. Furthermore, we could phrase the definition in terms of the function  $d(m)$ . That is, we could define the multiplicity of  $\Omega$  to be the least positive integer  $n$  such that  $N_1^{-1}(\mathfrak{D}) \cap \{m | d(m) > n\}$  has void interior.

Theorem 3.9  $N_1$  and  $N_2$  are unitarily equivalent iff  $N_1$  and  $N_2$  have the same spectral classes with the same respective multiplicities.

Proof: Suppose that  $U$  is a unitary operator and  $UN_1 = N_2U$ . The map  $\Phi(A) = UAU^{-1}$  is an isometric isomorphism from  $E^1$  to  $E^2$  with the property that  $\hat{A}(m) = \Phi(A)\hat{\Phi(m)}$  for all  $A \in E^1$  and  $m \in M_1$ .

Claim: The map  $A \rightarrow UAU^{-1}$  takes  $(E^1)'$  to  $(E^2)'$ .

If  $A \in (E^1)'$  and  $B \in E^2$ , then  $U^{-1}BU \in E^1$ . Hence,

$$U^{-1}BUA = AU^{-1}BU$$

$$BUAU^{-1} = AUA^{-1}B$$

Therefore,  $U(E^1)'U^{-1} \subset (E^2)'$ . Similarly,  $U^{-1}(E^2)'U \subset (E^1)'$ , and hence  $U(E^1)'U^{-1} = (E^2)'$ .

If  $\xi_0$  is cyclic for  $(E^1)'$ , then  $(E^1)'\xi_0 = H =$

$$U(E^1)'U^{-1}U\xi_0 = (E^2)'U\xi_0,$$

and consequently,  $U\xi_0$  is cyclic for  $(E^2)'$ .

Lemma 3.10 Suppose  $\{(K_i, \eta_i)\}$  is a canonical decomposition for  $E^1$ . If  $L_i = E^2U\eta_i$ , then  $\{(L_i, U\eta_i)\}$  is a canonical decomposition for  $E^2$ .

Proof:  $H = \sum_i \oplus UE^1U^{-1}U\eta_i = \sum_i \oplus E^2U\eta_i$ . We now need only prove that  $\phi_{U\eta_i} = \pi_{S_{U\eta_i}}$ . Suppose  $A \in E^2$ .

$$\begin{aligned} (AU\eta_i | U\eta_i) &= (U^{-1}AU\eta_i | \eta_i) \\ &= \int_{E_{\eta_i}} (U^{-1}AU)\hat{(m)} d\mu_1(m) \end{aligned}$$

$$\begin{aligned}
&= (P_{S_{\eta_1}} U^{-1} A U \eta_1 | \eta_1) \\
&= (U P_{S_{\eta_1}} U^{-1} A U \eta_1 | U \eta_1)
\end{aligned}$$

Claim:  $U P_{S_{\eta_1}} U^{-1} = P_{S_{U \eta_1}} \in E^2$ .

We show that  $U P_{S_{\eta_1}} = P_{S_{U \eta_1}} U$ . Let  $x \in H$ ,  $x = x_1 + x_2$  where

$P_{S_{\eta_1}} x_1 = x_1$  and  $P_{S_{\eta_1}} x_2 = 0$ . Then  $U P_{S_{\eta_1}} x = U x_1$ .

$$P_{S_{U \eta_1}} U x = P_{S_{U \eta_1}} U x_1 + P_{S_{U \eta_1}} U x_2.$$

Suppose  $B \in (E^2)'$ . Then  $(B U \eta_1 | U x_2) = (U^{-1} B U \eta_1 | x_2) = 0$  because  $U^{-1} B U \in (E^1)'$ . Hence  $P_{S_{U \eta_1}} U x_2 = 0$ . There exist

$A_n \in (E^1)'$  such that  $A_n \eta_1 \rightarrow x_1$ ,  $U A_n \eta_1 \rightarrow U x_1$ ,

$$U A_n U^{-1} U \eta_1 \rightarrow U x_1,$$

and therefore  $P_{S_{U \eta_1}} U x_1 = U x_1$ . Hence,  $U P_{S_{\eta_1}} U^{-1} = P_{S_{U \eta_1}}$

and  $(A U \eta_1 | U \eta_1) = (P_{S_{U \eta_1}} A U \eta_1 | U \eta_1)$  which implies that

$$\phi_{U \eta_1} = \pi_{S_{U \eta_1}}.$$

Now if  $\mathfrak{D}$  is a measurable set of complex numbers, then

$$\{\phi(m) | m \in \hat{N}_1^{-1}(\mathfrak{D}) \cap S_{\eta_1}\} = \hat{N}_2^{-1}(\mathfrak{D}) \cap S_{U \eta_1}.$$

Therefore, if  $N_1$  and  $N_2$  are unitarily equivalent, then they have the same spectral classes with the same respective multiplicities.

Suppose that  $N_1$  and  $N_2$  have the same spectral classes with the same respective multiplicities. Let  $\{(K_1, \eta_1)\}$

be a canonical decomposition for  $E^1$ , and let  $\{(L_i, \xi_i)\}$  be a canonical decomposition system for  $E^2$ .  $E^1|_{K_1}$  is a maximal commutative symmetric subring of  $B(K_1)$ ; and, likewise,  $E^2|_{L_1}$  is a maximal commutative symmetric subring of  $B(L_1)$ . The maximal ideal space of  $E^1|_{K_1}$  is  $S_{\eta_1} \subseteq M_1$ , and the maximal ideal space of  $E^2|_{L_1}$  is  $S_{\xi_1} \subseteq M_2$ . Our hypothesis tells us that for each measurable subset  $\mathfrak{D}$  of the complex plane,  $N_1^{-1}(\mathfrak{D}) \cap S_{\eta_1}$  has void interior iff  $N_2^{-1}(\mathfrak{D}) \cap S_{\xi_1}$  has void interior. Hence, by Theorem 3.6, there exists an isometry  $V_1: K_1 \rightarrow L_1$  such that  $V_1 N_1 = N_2 V_1$ . Let  $U = \sum_1 V_1$ . Then  $U$  is a unitary operator, and  $U N_1 = N_2 U$ .

The question arises, "What is the relation between spectral classes and eigenvalues and the corresponding multiplicities of each?" Suppose that  $N$  is a normal operator which generates the symmetric ring  $E$  with maximal ideal space  $M$ . If  $\{\lambda_0\}$  is not equivalent to  $\emptyset$ , then there exists a point  $m_0 \in M$  such that  $m_0$  is isolated and  $N^{\wedge}(m_0) = \lambda_0$ . In this case  $\lambda_0$  is an eigenvalue of  $N$ , and the multiplicity as we have defined it of the spectral class of which  $\{\lambda_0\}$  is a representative is equal to the ordinary multiplicity of the eigenvalue  $\lambda_0$ .

If  $E$  is a commutative, symmetric, weakly closed ring with maximal ideal space  $M$ , then for  $m \in M$ , the functional  $f_m(A) = A^{\wedge}(m)$  is a norm continuous, positive functional on the ring  $E$ . It is not, however, in general weakly



continuous. The functional  $f_{m_0}$  will be weakly continuous iff  $m_0$  is isolated, in which case  $\hat{A}(m_0)$  will be an eigenvalue for  $A \in E$ . One might wonder if a geometric characterization of a normal operator can be given in terms of the weakly continuous functionals on the weakly closed, symmetric ring generated by the operator. Theorems 3.11 and 3.12 answer this question in the affirmative.

Theorem 3.11 Suppose that  $E^1$  is maximal with cyclic vector  $\xi_1$  to which there corresponds the measure  $\mu_1$  on  $M_1$ . Suppose that there exists an isometric isomorphism  $\phi: E^1_* \rightarrow E^2_*$  such that  $\phi f \geq 0$  iff  $f \geq 0$  and  $(\phi f)(N_2) = f(N_1)$  for all  $f$ . Then  $N_1$  and  $N_2$  are unitarily equivalent.

Proof: Now we can consider  $E^1$  to be  $(E^1_*)^*$ . We define a map  $\theta: E^2 \rightarrow E^1$  by  $[\theta(A)](f) = A(\phi f)$  for all  $f \in E^1_*$ ,  $A \in E^2$ .

$$\begin{aligned} ||\theta(A)|| &= \sup |[\theta(A)](f)| = \sup_{|f|=1} |A(\phi f)| \\ &= \sup_{|g|=1} |A(g)| = ||A||. \end{aligned}$$

We see that  $\theta(A)$  is bounded and therefore in  $E^1$  as was claimed. In fact,  $\theta$  is an isometry in the operator (or, equivalently,  $L_\infty$ ) norm. Also,  $[\theta(N_2)](f) = N_2(\phi f) = (\phi f)(N_2) = f(N_1) = N_1(f)$ .

Hence,  $\theta(N_2) = N_1$ .

If  $A \in E^2$ ,  $A \geq 0$ ,  $f \in E^1_*$ , and  $f \geq 0$ , then

$$\theta(A)f = A(\phi f) = (\phi f)(A) \geq 0.$$

Consequently,  $\theta(A) \geq 0$ . Similarly, if  $\theta(A) \geq 0$ , then  $A \geq 0$ .

Claim: If  $A \geq 0$  and  $B \geq 0$ , then  $\theta(A \wedge B) = \theta(A) \wedge \theta(B)$ .

Proof of claim: Since  $A \wedge B \leq A$ ,  $\theta(A \wedge B) \leq \theta(A)$ . Likewise,  $\theta(A \wedge B) \leq \theta(B)$ . Therefore,  $\theta(A \wedge B) \leq \theta(A) \wedge \theta(B)$ . Now there exists an element  $C \in E^2$  such that  $\theta(C) = \theta(A) \wedge \theta(B)$ . But  $\theta(C) \leq \theta(A)$  implies  $C \leq A$ , and  $\theta(C) \leq \theta(B)$  implies  $C \leq B$ . Hence  $C \leq A \wedge B$  and  $\theta(C) = \theta(A) \wedge \theta(B) \leq \theta(A \wedge B)$ . Thus  $\theta(A) \wedge \theta(B) = \theta(A \wedge B)$ . Note that if  $A$  and  $B$  are projections in  $E^2$ , then  $A \wedge B = AB$  and  $\theta(A) \wedge \theta(B) = \theta(AB)$ .

Suppose that  $A$  and  $B$  are projections in  $E^2$  such that  $AB \neq 0$ . Since  $\|\theta(AB)^\wedge(m)\|_\infty = 1$ , there exists a point  $m_0 \in M_1$  such that  $\theta(A)^\wedge(m_0) = 1 = \theta(B)^\wedge(m_0)$ . Hence  $\|\theta(A)\theta(B)\|_\infty = 1$ . From  $0 \leq \theta(A)^\wedge(m) \leq 1$  and

$$0 \leq \theta(B)^\wedge(m) \leq 1$$

we see that  $\theta(A)\theta(B) \leq \theta(A)$  and likewise  $\theta(A)\theta(B) \leq \theta(B)$ . Therefore,  $\theta(A)\theta(B) \leq \theta(A) \wedge \theta(B) = \theta(AB)$ .

Suppose that  $\theta(A)\theta(B) = \theta(C) < \theta(AB)$ . We can pick a projection  $D \in E^2$  such that  $D < AB$  and  $\|CD\|_\infty < 1$ . Then  $\theta(A)\theta(B)\theta(D) = \theta(C)\theta(D) \leq \theta(C) \wedge \theta(D) = \theta(C \wedge D)$ . Consequently,  $\|\theta(C \wedge D)\|_\infty = \|C \wedge D\|_\infty = \|CD\|_\infty < 1$ . But the argument given above to demonstrate that  $\|\theta(A)\theta(B)\|_\infty = 1$  will suffice to show that  $\|\theta(A)\theta(B)\theta(D)\|_\infty = 1 = \|\theta(C)\theta(D)\|_\infty$  which is a contradiction. Therefore,  $\theta(A)\theta(B) = \theta(AB)$ . By linearity,  $\theta$  is multiplicative on all clopen step functions. But  $\theta$  is an isometry in  $L_\infty$  norm so that

$$\theta(A)\theta(B) = \theta(AB)$$

for all  $A, B, \in E^2$ .

We define a measure  $\alpha$  on  $M_2$  as follows. Suppose  $U$  is a clopen subset of  $M_2$ . We let  $\alpha(U) = \int_{M_1} \theta(P_U)^{\wedge}(m) d\mu_1(m)$ . If  $K$  is measurable with respect to  $\mu_2$ , then there exists a clopen set  $V$  such that  $K \approx V$ . We define  $\alpha(K) = \alpha(V)$ .  $\alpha$  is positive because  $\theta$  takes positive functions to positive functions. Furthermore,  $\alpha$  is supported on all of  $M_2$ , and  $\alpha$  is absolutely continuous with respect to  $\mu_2$ . Therefore, there exists a function  $\zeta(m) \in L_1(M_2, \mu_2)$  such that

$$\int_{M_2} A^{\wedge}(m) d\alpha(m) = \int_{M_2} A^{\wedge}(m) \zeta(m) d\mu_2(m)$$

for all  $A \in E^2$ . Note that  $\zeta(m) \geq 0$ .

There exist operators  $A_n \in E^2$  such that

$$0 \leq A_n^{\wedge}(m) \leq A_{n+1}^{\wedge}(m)$$

and  $\|A_n^{\wedge}(m) - \zeta(m)\|_1 \rightarrow 0$ . Let  $B_n$  denote the unique positive definite operator such that  $B_n^2 = A_n$ . Then  $B_n \in E^2$  and if  $i > j$ ,

$$\begin{aligned} \|(B_i - B_j)\xi_2\|^2 &= ([B_i - B_j]\xi_2 | [B_i - B_j]\xi_2) \\ &\leq \int_{M_2} [A_i^{\wedge}(m) - B_i^{\wedge}(m)B_j^{\wedge}(m)] d\mu_2(m) \\ &\quad + \int_{M_2} [B_i^{\wedge}(m)B_j^{\wedge}(m) - A_j^{\wedge}(m)] d\mu_2(m) \\ &\leq 2 \int_{M_2} [A_i^{\wedge}(m) - A_j^{\wedge}(m)] d\mu_2(m). \end{aligned}$$

Since  $A_n^{\wedge}(m)$  is Cauchy in  $L_1$  norm,  $B_n \xi_2$  is Cauchy and converges to some vector  $\xi'_2 \in H$ .

$$\begin{aligned}
\int_{M_2} \hat{A}(m) \, d\alpha(m) &= \int_{M_2} \hat{A}(m) \zeta(m) \, d\mu_2(m) \\
&= \lim_n (AA_n \xi_2 | \xi_2) = \lim_n (AB_n \xi_2 | B_n \xi_2) \\
&= (A \xi'_2 | \xi'_2).
\end{aligned}$$

If  $I$  is the identity operator, the  $(\phi f)(I) = I(\phi f) = \theta(I)(f) = f(\theta(I))$ .

Since  $\phi$  is an isometry,  $(\phi f)(I) = ||f|| = f(\theta(I))$  for all positive functionals  $f \in E^1_*$ . Hence,  $\theta(I) = I$ , and  $|\xi'_2| = 1$ .

We define an operator  $U$  by  $U\theta(A)\xi_1 = A\xi'_2$  for all  $A \in E^2$ .  $|U\theta(A)\xi_1|^2 = |A\xi'_2|^2 = (A^*A\xi'_2 | \xi'_2)$

$$\begin{aligned}
&= \int_{M_2} (\hat{A^*A})(m) \, d\alpha(m) \\
&= \int_{M_1} \theta(\hat{A^*A})(m) \, d\mu_1(m) \\
&= (\theta(A)^*\theta(A)\xi_1 | \xi_1) = |\theta(A)\xi_1|^2.
\end{aligned}$$

Therefore  $U$  can be extended to a unitary operator. Now if  $A \in E^2$ ,  $UN_1\theta(A)\xi_1 = U\theta(N_2)\theta(A)\xi_1 = U\theta(N_2A)\xi_1$

$$= N_2A\xi'_2 = N_2U\theta(A)\xi_1.$$

Consequently  $UN_1 = N_2U$  on a dense subset of  $H$  which implies that  $UN_1 = N_2U$  everywhere. Therefore,  $N_1$  and  $N_2$  are unitarily equivalent. Note that  $U^{-1}AU = \theta(A)$ .

We are once again confronted with the problem of extending our results to the case that  $E^1$  and  $E^2$  are not maximal, and again we shall develop a theory of spectral multiplicity based on a canonical decomposition system.

First of all, we note that in Theorem 3.11 we really

needed only to assume that  $\Phi$  was an isometric isomorphism from the cone of positive functionals in  $E_*^1$  to the cone of positive functionals in  $E_*^2$ , for such a map could be extended to an isometric isomorphism of  $E_*^1$  to  $E_*^2$ . We denote the positive functionals of  $E_*^1$  by  $(E_*^1)^+$ .

If  $E$  is a commutative, symmetric ring containing the identity, and if  $\xi_0$  is a unit cyclic vector for  $E$ , then every positive functional in  $E_*$  has the form

$$T_\zeta(A) = \int_M A(m) \zeta(m) d\mu(m)$$

where  $M$  is the maximal ideal space of  $E$ ,  $\mu$  is the measure of  $M$  corresponding to  $\xi_0$ , and  $\zeta(m) \in L_1(M, \mu)$  is a continuous function from  $M$  to the non-negative extended reals. Suppose  $\{(K_i, S_{\eta_i})\}$  is a canonical decomposition for  $E$ . Then we define the multiplicity of  $T_\zeta$  to be the least positive integer  $n$  such that  $\{m | \zeta(m) > 0\} \cap S_{\eta_{n+1}} = \emptyset$  provided such an  $n$  exists. Otherwise we shall say that  $T_\zeta$  has infinite multiplicity. The remarks following the definition of multiplicity of spectral classes are equally applicable here.

Theorem 3.12  $N_1$  is unitarily equivalent to  $N_2$  iff there exists an isometric isomorphism  $\Phi: (E_*^1)^+ \rightarrow (E_*^2)^+$  such that  $(\Phi f)(N_2) = f(N_1)$  and the multiplicity of  $\Phi(f) =$  the multiplicity of  $f$  for all  $f \in (E_*^1)^+$ .

Proof: If there exists a unitary operator  $U$  such that

$UN_1 = N_2U$ , then the map  $\Phi$  defined by  $(\Phi f)(A) = f(U^{-1}AU)$  for all  $A \in E^2$  satisfies the above requirements.

Suppose that there exists such a map  $\Phi$ . Let  $\{(K_i, \eta_i)\}$  be a canonical decomposition system for  $E^1$ , and let  $\{(L_i, \xi_i)\}$  be a canonical decomposition system for  $E^2$ . The maximal ideal space of  $E^1|_{K_i}$  is  $S_{\eta_i}$ , and the maximal ideal space of  $E^2|_{L_i}$  is  $S_{\xi_i}$ . Furthermore,  $E^1|_{K_i}$  is maximal in  $B(K_i)$ , and  $E^2|_{L_i}$  is maximal in  $B(L_i)$ . The positive functionals in  $(E^1|_{K_i})_*$  arise from non-negative functions in  $L_1(M_1, \mu_1)$  whose supports are contained in  $S_{\eta_i}$ . Likewise, the positive functionals in  $(E^2|_{L_i})_*$  arise from non-negative functions in  $L_1(M_2, \mu_2)$  whose supports are contained in  $S_{\xi_i}$ . Our hypothesis guarantees that there exists  $\Phi_i$  a restriction of  $\Phi$  such that  $\Phi_i$  is an isometric isomorphism from  $(E^1|_{K_i})_*^+$  to  $(E^2|_{L_i})_*^+$  where

$$\Phi_i(N_1|_{K_i}) = N_2|_{L_i}.$$

Therefore, by Theorem 3.11, there exist unitary operators  $V_i: K_i \rightarrow L_i$  such that  $V_i N_1 = N_2 V_i$ . Let  $U = \sum_i V_i$ . Then  $UN_1 = N_2U$  and  $U$  is unitary.

The salient restriction on the normal operators in Theorems 3.10 and 3.12 is the requirement that they operate on a separable space. This hypothesis is necessary to insure that  $(E^1)'$  has a cyclic vector, which is essential, in light of the extensive use of measure theoretic methods

in the above proofs. The following example suggests that the methods that we employed above would have to be modified to handle the non-separable case.

Example 3.13: Let  $\{e_\lambda\}$  be an orthonormal basis for a Hilbert space  $H$  indexed by the set of complex numbers of modulus 1. We define a unitary operator  $U$  by defining it on the basis as  $Ue_\lambda = \lambda e_\lambda$ . Suppose  $A \in B(H)$  and  $AU = UA$ .

If  $Ae_\lambda = \sum_{i=1} a_i e_{\lambda_i}$ , then  $UAe_\lambda = \sum_{i=1} a_i \lambda_i e_{\lambda_i}$ .

$AUe_\lambda = A(\lambda e_\lambda) = \lambda Ae_\lambda = \sum_i a_i \lambda e_{\lambda_i}$ . Therefore,

$a_i \lambda_i = a_i \lambda$  for each  $i$ . Hence  $a_i = 0$  when  $\lambda \neq \lambda_i$ .

Therefore,  $Ae_\lambda = ae_\lambda$  for some  $a$ . The ring of all operators which commute with  $U$  is exactly the ring generated by the collection of one-dimensional projections which project on the subspaces generated by the  $e_\lambda$ 's. This ring does not have a cyclic vector.

We now wish to relate our theory to a previous theory of spectral multiplicity as explicated in [2], Introduction to Hilbert Space and the Theory of Spectral Multiplicity by P. R. Halmos. The two theories do not seem to be directly comparable. Both, of course, associate with each normal operator  $N$  a multiplicity function which completely characterizes the operator; but the domains of definition of the multiplicity functions of the two theories are not

the same. We have assigned a multiplicity to each weakly continuous positive functional defined on the weakly closed ring generated by  $N$  and  $N^*$ . In the Halmos theory, a multiplicity is defined for each norm continuous positive functional defined on the norm closed ring generated by  $N$  and  $N^*$ . Furthermore, the Halmos theory is not burdened with the restriction of separability.

Suppose  $N$  is a normal operator on a separable Hilbert space  $H$ . Let  $E$  denote the weakly closed ring generated by  $N$  and  $N^*$ , and let  $M$  denote the maximal ideal space of  $E$ . We shall assume that  $E$  contains the identity. Let  $\{(K_i, \eta_i)\}$  denote a canonical decomposition system for  $E$ . Recall that the multiplicity we defined was given in terms of clopen subsets of  $M$  (i.e. projections in  $E$ ) and a canonical decomposition system. Although the Halmos theory does not utilize the canonical decomposition system, we can give a simple, equivalent description of the theory using these concepts. With the two multiplicities defined in similar language, the connection between them should be clear.

In the Halmos theory, a multiplicity is assigned to each projection in  $E$  as follows. If  $P = 0$ , then the multiplicity of  $P$  is 0. If  $P \in E$ ,  $P \neq 0$ , then  $P = P_U$  for some nonempty clopen set  $U$  contained in  $M$ . The multiplicity of  $P$  is the largest integer  $n$  (provided such exists) such that  $U \cap S_{\eta_n} = U$ . If  $U \cap S_{\eta} = U$  for all  $n$ ,



then  $P$  is said to have infinite multiplicity.

Suppose  $x \in H$ . Consider the measure  $\alpha_x(\mathfrak{D}) = (P(\mathfrak{D})x|x)$  where  $P(\mathfrak{D})$  denotes the spectral decomposition of  $N$ . If  $\mu$  is a measure on the spectrum of  $N$ , then  $\{x | \alpha_x < < \mu\}$  is a subspace invariant under  $E'$ . The projection on this subspace is therefore in  $E$ . The multiplicity of  $\mu$  is defined to be the multiplicity of that projection. If  $\mu = \alpha_x$  for some  $x$ , then the corresponding projection is just  $P_{S_x}$ .

The point of contact between the two theories is the relationship of the respective multiplicity functions to clopen subsets of  $M$ . Essentially, we defined a multiplicity for clopen sets that agrees with the Halmos definition for clopen sets  $U$  with the property that  $U \cap S_{\eta_n} = U$  or  $\emptyset$  for all  $n$ . For other clopen sets, the two multiplicities differ. The multiplicity which we defined is increasing, whereas the Halmos multiplicity is decreasing. That is, if we denote the multiplicity defined in this paper by  $m_1$  and denote the Halmos multiplicity by  $m_2$ , then for projections  $P$  and  $Q$  in  $E$  with  $P \leq Q$ , we have  $m_1(P) \leq m_1(Q)$  and  $m_2(Q) \leq m_2(P)$ .

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## BIOGRAPHY

Robert Russell Butts was born in Wichita Falls, Texas, on July 22, 1943. He attended the public schools of Iowa Park, Texas, and graduated in May, 1961. From September, 1961, to August, 1964, he attended North Texas State University where he received a B.A.

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## EXAMINATION AND THESIS REPORT


**Candidate:** Robert R. Butts

**Major Field:** Mathematics-Analysis


**Title of Thesis:** COMMUTATIVE SYMMETRIC RINGS AND THEIR APPLICATIONS TO  
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**Approved:**

  
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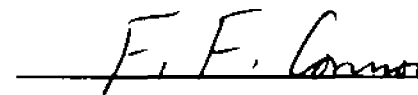
  
Dean of the Graduate School

### EXAMINING COMMITTEE:









**Date of Examination:**

May 6, 1969